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KASR TARTIBLI OPERATORLAR BOSHLANG'ICH TUSHUNCHALAR VA ABEL INTEGRAL TENGLAMASI YECHIMLARI

Annotatsiya: Ushu ishda kasr tartibli operatorlar va Abel integral tenglamalari xususan murakkab modellarni yechishda, materiallarni va harakatni modellovchi tenglamalarni yechishda qo'llaniladi. Abel integral tenglamasni, kasr tartibli operatorlar xossalari foydalanib, kasr tartibli integrallar va kasr tartibli hosilalar yoramida masala yechimi keltirilgan.

Kalit so'zlar: Kasr tartibli operatorlar, kasr tartibli hosila, kasr tartibli integral.

ДРОБНЫЕ ОПЕРАТОРЫ. ИСХОДНЫЕ ПОНЯТИЯ И РЕШЕНИЯ ИНТЕГРАЛЬНОГО УРАВНЕНИЯ АБЕЛЯ

Аннотация: В данной работе операторы дробного порядка и интегральные уравнения Абеля используются, в частности, при решении сложных моделей, при решении уравнений, моделирующих материалы и движение. Используя интегральное уравнение Абеля, свойства дробных операторов, решение задачи представлено с помощью дробных интегралов и дробных производных

Ключевые слова: Дробные операторы, дробная производная, дробный интеграл.

FRACTIONAL OPERATORS INITIAL CONCEPTS AND SOLUTIONS OF ABEL'S INTEGRAL EQUATION

Abstract: In this work, fractional order operators and Abel's integral equations are used, in particular, in solving complex models, in solving equations modeling materials and motion. Using the Abel integral equation, the properties of fractional operators, the solution of the problem is presented with the help of fractional integrals and fractional derivatives.

Keywords: Fractional operators, fractional derivative, fractional integral.

Ushbu

$$\frac{1}{\Gamma(\alpha)}\int_{\alpha}^{x}\frac{\varphi(t)dt}{(x-t)^{1-\alpha}}=f(x), \qquad 0<\alpha<1 \tag{1}$$

ko'rinishdagi integral tenglama Abel integral tenglamasi deyiladi[7].

(1) tenglama quydagi usulda yechiadi. Bu tenglamada x ni t bilan, t ni s bilan amashtirib, so'ngra tenglamaning xar ikki tomonini $(x-t)^{-\alpha}$ ifodaga ko'paytiramiz va t bo'yicha a dan x gacha integrallaymiz:

$$\int_a^x \frac{dt}{(x-t)^{\alpha}} \int_a^t \frac{\varphi(s)ds}{(t-s)^{1-\alpha}} = \Gamma(\alpha)\int_a^x \frac{f(t)dt}{(x-t)^{\alpha}}.$$

Dirixle formulasiga ko‘ra integrallash tartibini almashtirib,

$$\int_a^x \varphi(s)ds \int_s^x \frac{dt}{(x-t)^{\alpha}(t-s)^{1-\alpha}} = \Gamma(\alpha)\int_a^x \frac{f(t)dt}{(x-t)^{\alpha}} \qquad (2)$$

tenglikni hosil qilamiz. Tenglikni chap tomonidagi ichki integralda $t = s + \tau(x-s)$ almashtirish bajarsak,

$$\int_s^x (x-t)^{-\alpha}(t-s)^{\alpha-1}dt = \int_0^1 \tau^{\alpha-1}(1-\tau)^{-\alpha}d\tau = B(\alpha,1-\alpha) = \Gamma(\alpha)\Gamma(1-\alpha)$$

tenglik kelib chiqadi. U holda, (2) ga asosan

$$\int_a^x \varphi(s)ds = \frac{1}{\Gamma(1-\alpha)}\int_a^x \frac{f(t)dt}{(x-t)^{\alpha}}. \qquad (3)$$

Bu tenglikning har ikki tomonini differensiallab, Abel integral tenglamasining yechimini hosil qilamiz[1]:

$$\varphi(s) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(t)dt}{(x-t)^{\alpha}}. \qquad (4)$$

Shunday qilib, agar (1) tenglamaning yechimi mavjud bo‘lsa, u (4) ko‘rinishda ifodalanar ekan. Bu formulani hosil qilish jarayonidan kelib chiqadiki, agar yechim mavjud bo‘lsa, u yagona.

Shu usulda ko‘rsatish mumkinki, ushbu

$$\frac{1}{\Gamma(\alpha)}\int_x^b \frac{\varphi(t)dt}{(t-x)^{1-\alpha}} = f(x), \qquad 0<\alpha<1 \qquad (5)$$

integral tenglamani yechimi

$$\varphi(x) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{f(t)dt}{(t-x)^{\alpha}} \qquad (6)$$

formula bilan aniqlanadi.

Matematik analiz kursidan ma’lumki, n – karrali integral uchun quyidagi formula o‘rinli:

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$$\int_a^{x_0} dx_1 \int_a^{x_1} dx_2 \ldots \int_a^{x_{n-1}} \varphi(t)\,dt = \frac{1}{(n-1)!}\int_a^{x_0} (x_0 - t)^{n-1}\varphi(t)\,dt, \quad n \quad N. \tag{7}$$

$(n-1)! = \Gamma(n)$ ekanligini e'tiborga olib, (7) tenglikning o'ng tomonini n ning kasr qiymatlari uchun ham aniqlash mumkin[2].

(7) tenglikka mos ravishd kasr tartibli integrallarni quyidagi tartibda aniqlaymiz.

Ta'rif. $\varphi(x) \quad L_1(a,b) \quad (a < b < + \quad)$ bo'lsin. Ushbu

$$D_{xb}^{-\alpha}\varphi(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\varphi(t)\,dt, \quad \alpha > 0 \tag{8}$$

ko'rinishdagi ifodalar $\varphi(x)$ funksiyaning α (kasr) tartibli (Riman-Liuvill ma'nosida) integarallari deyiladi.

$D_{ax}^{-\alpha}\varphi(x)$ va $D_{xb}^{-\alpha}\varphi(x)$ funksiyalar (a,b) oraliqning deyarli barcha nuqtalarida aniqlangan bo'lib, $L_1(a,b)$ sinfga tegishli bo'ladi.

Bu ta'rifga asosan (1) va (5) Abel integral tenglamalarini

$$D_{ax}^{-\alpha}\varphi(x) = f(x), \qquad D_{xb}^{-\alpha}\varphi(x) = f(x) \tag{9}$$

ko'rinishda yozish mumkin.

Agar $0 < \alpha_1,\ \alpha_2 < +$ bo'lsa, deyarli hamma $x \quad (a,b)$ uchun

$$D_{ax}^{-\alpha_2}D_{ax}^{-\alpha_1}f(x) = D_{ax}^{-\alpha_1}D_{ax}^{-\alpha_2}f(x) = D_{ax}^{-(\alpha_1+\alpha_2)}f(x) \tag{10}$$

tenglik o'rinli bo'ladi. Haqiqatan ham,

$$D_{ax}^{-\alpha_2}D_{ax}^{-\alpha_1}f(x) = \frac{1}{\Gamma(\alpha_1)}D_{ax}^{-\alpha_2}\int_a^x (x-s)^{\alpha_1-1} f(s)\,ds =$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_a^x \int_a^t (t-s)^{\alpha_1-1} f(s)\,ds\ (x-t)^{\alpha_2-1}\,dt =$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_a^x f(s)\,ds \int_s^x (x-t)^{\alpha_2-1}(t-s)^{\alpha_1-1}\,dt.$$

Oxirgi ichki integralda $t = s + (x - s)\tau$ almashtirish bajarish natijasida quyidagi tenglikni hosil qilamiz[1]:

$$\int_s^x (x-t)^{\alpha_2-1}(t-s)^{\alpha_1-1}\,ds = (x-s)^{\alpha_1+\alpha_2-1}\int_0^1 \tau^{\alpha_1-1}(1-\tau)^{\alpha_2-1}\,d\tau =$$

$$= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}(x-s)^{\alpha_1+\alpha_2-1}$$

Bu esa (10) tenglikning to‘g‘riligini ko‘rsatadi.

Ta’rifga asosan,

$$D_{ax}^{0} f(x) = f(x) \qquad (11)$$

deb hisoblaymiz.

Ta’rif. $\varphi(x)$ funksiya $[a,b]$kesmada aniqlangan bo‘lsin.

$$D_{ax}^{\alpha}\varphi(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{\varphi(t)dt}{(x-t)^{\alpha}}, \qquad 0<\alpha<1, \qquad (12)$$

$$D_{xb}^{\alpha}\varphi(x) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{\varphi(t)dt}{(t-x)^{\alpha}}, \qquad 0<\alpha<1 \qquad (13)$$

ko‘rinishdagi ifodalar $\varphi(x)$ funksiyaning α (kasr) tartibli (Liuvill ma’nosidagi) hosilalari deyiladi.

Bu ta’rifga asosan (1) va (5) Abel integral tenglamalari yechimlarini beruvchi (4) va (6) tengliklari mos ravishda

$$\varphi(x) = D_{ax}^{\alpha} f(x), \qquad \varphi(x) = D_{xb}^{\alpha} f(x) \qquad (14)$$

ko‘rinishda yozish mumkin.

Eslatib o‘tamizki, kasr tartibli integrallar $\alpha > 0$ tartibgacha aniqlangan. Lekin (12), (13) kasr tartibli hosilalar faqatgina $0 < \alpha < 1$ bo‘lganda aniqlangan. Kasr tartiblib hosilalarni $\alpha \quad 1$ bo‘lganda aniqlashga o‘tishdan oldin kasr tartibli hosilalar mavjudligining yetarli shartini keltiramiz.

Lemma. Agar $\varphi(x)$ funksiya $[a,b]$ kesmada absolyut uzuluksiz bo‘lsa, $[a,b]$ kesmaning deyarli barcha nuqtalarida $\varphi(x)$ funksiyaning kasr tartibli hosilalari mavjud bo‘lib, quyidagi formulalar o‘rinli bo‘ladi:

$$D_{ax}^{\alpha}\varphi(x)=\frac{1}{\Gamma(1-\alpha)}\left[\frac{\varphi(a)}{(x-a)^{\alpha}}+\int_{a}^{x}\frac{\varphi'(t)dt}{(x-t)^{\alpha}}\right], \qquad 0<\alpha<1,$$

$$D_{xb}^{\alpha}\varphi(x)=\frac{1}{\Gamma(1-\alpha)}\left[\frac{\varphi(b)}{(b-x)^{\alpha}}+\int_{x}^{b}\frac{\varphi'(t)dt}{(t-x)^{\alpha}}\right], \qquad 0<\alpha<1.$$

Misol. $\varphi(x)=(x-a)^{\alpha-1}$ bo‘lsin. U holda, (1.1.13) tenglikka asosan,

$$D_{ax}^{\alpha}\varphi(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{a}^{x}(x-t)^{-\alpha}(t-a)^{\alpha-1}dt.$$

Integral o‘zgaruvchisini $t=a+(x-a)z$ formula bilan almashtirsak,

$$D_{ax}^{\alpha}\varphi(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{0}^{1}z^{\alpha-1}(1-z)^{-\alpha}dz=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}B(\alpha,1-\alpha)=0$$

tenglik kelib chiqadi. Demak , $\varphi(x)=(x-a)^{\alpha-1}$ funksiya $\alpha \in (0,1)$ tartibli hosila uchun o‘zgarmas son vazifasini bajaradi.

Endi $\alpha \geq 1$ bo‘lib, $[\alpha]$ - uning butun qismi, $\{\alpha\}$ - esa kasr qismi bo‘lsin. Agar α - butun son bo‘lsa, α tartibli hosilslar sinifida oddiy hosilalarni olamiz:

$$D_{ax}^{\alpha}=\left(\frac{d}{dx}\right)^{\alpha}, \qquad D_{xb}^{\alpha}=\left(-\frac{d}{dx}\right)^{\alpha}, \qquad \alpha=1,2,3,\ldots .$$

Agar α - butun son bo‘lmasa, α tartibli hosilalarni quyidagicha aniqlaymiz:

$$D_{ax}^{\alpha}\varphi(x)=\left(\frac{d}{dx}\right)^{[\alpha]}D_{ax}^{\{\alpha\}}\varphi(x)=\left(\frac{d}{dx}\right)^{[\alpha]+1}D_{ax}^{\{\alpha\}-1}\varphi(x),$$

$$D_{xb}^{\alpha}\varphi(x)=\left(-\frac{d}{dx}\right)^{[\alpha]}D_{xb}^{\{\alpha\}}\varphi(x)=\left(-\frac{d}{dx}\right)^{[\alpha]+1}D_{xb}^{\{\alpha\}-1}\varphi(x).$$

Demak, umumiy holda, $\alpha \geq 1$ bo‘lganda

$$D_{ax}^{\alpha}\varphi(x)= \frac{d}{dx}^{\,n} D_{ax}^{\alpha-n}\varphi(x), \qquad n=[\alpha]+1, \tag{15}$$

$$D_{xb}^{\alpha}\varphi(x)=(-1)^{n} \frac{d}{dx}^{\,n} D_{xb}^{\alpha-n}\varphi(x), \qquad n=[\alpha]+1. \tag{16}$$

Odatda α $(\alpha>0)$ kasr tartibli integrallar ko‘rinishida ifodalanuvchi funksiyalar sinfi $D_{ax}^{-\alpha}(L_p)$ bilan belgilanadi, ya’ni

$$D_{ax}^{-\alpha}(L_p)=\{f(x): f(x)=D_{ax}^{-\alpha}\varphi(x), \quad \varphi(x) \quad L_p(a,b), \ 1 \quad p< \quad \}.$$

Quyidagi teorema o‘rinli.

Teorema. $\alpha>0$ bo‘lsin. U holda

$$D_{ax}^{\alpha}D_{ax}^{-\alpha}\varphi(x)=\varphi(x), \qquad D_{xb}^{\alpha}D_{xb}^{-\alpha}\varphi(x)=\varphi(x) \tag{17}$$

tengliklar barcha $\varphi(x) \quad L_1(a,b)$ funksiyalar uchun,

$$D_{ax}^{-\alpha}D_{ax}^{\alpha}\varphi(x)=\varphi(x), \qquad D_{xb}^{-\alpha}D_{xb}^{\alpha}\varphi(x)=\varphi(x) \tag{18}$$

tengliklar esa mos ravishda barcha

$$\varphi(x) \quad D_{ax}^{-\alpha}(L_1), \qquad \varphi(x) \quad D_{xb}^{-\alpha}(L_1)$$

funksiyalar uchun bajariladi.

Agar oxirgi shartlar o‘rniga $\varphi(x) \quad L_1(a,b)$ bo‘lsa, (18) tengliklar umuman olganda noto‘g‘ri bo‘ladi va masalan, birinchisi quyidagi formula bilan almashadi$[3]$.

$$D_{ax}^{-\alpha}D_{ax}^{\alpha}\varphi(x)=\varphi(x)-\sum_{k=0}^{n-1}\frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)}\varphi_{n-\alpha}^{(n-k-1)}(a),$$

bu yerda $n=[\alpha]+1, \quad \varphi_{n-a}(x)=D_{ax}^{\alpha-n}\varphi(x).$

Demak, Abel integral tenglamalarini va ularning yechimlarini ifodalovchi (9) va (14) tengliklar bilan aniqlangan $f(x)$ va $\varphi(x)$ funksiyalarni mos ravishda (14) va (9) tengliklarga qo‘yish uchun yuqoridagi teorema shartini bajarilishi zarur bo‘ladi.

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