ISSN: 2181-4341, IMPACT FACTOR (RESEARCH BIB) – 7,245, SJIF – 5,431

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ELEMENTAR SIMMETRIK KO‘PHADLAR. RATSIONAL IFODALAR VA ULARNI SODDALASHTIRISH.

Annotatsiya: Ushbu maqolada Elementar simmetrik ko‘phadlar. Ratsional ifodalar va ularni soddalashtirish tushintiriladi.

Kalit so‘zlar: Elementar, simmetrik, ko‘phad, to‘plam, daraja.

Аннотация: В статье рассматриваются элементарные симметричные многочлены. Объясняются рациональные выражения и их упрощение.

Ключевые слова: Элементарный, симметричный, полиномиальный, множество, уровень.

Abstract: Elementary symmetric polynomials are discussed in this article. Rational expressions and their simplification are explained.

Key words: Elementary symmetric, polynomial, set, degree.

Elementar simmetrik ko‘phadlar. Ratsional ifodalar va ularni soddalashtirish.

Ta'rif 1. Noma’lumlarning o’rinlarini almashtirish bilan o‘zgarmaydigan bir nechta o‘zgaruvchili ko‘phad simmetrik ko‘phad deyiladi.
Misol: $f(x_1,x_2,x_3)=x_1+x_2+x_3$, chunki $f(x_1,x_2,x_3)=x_1+x_2+x_3$ o‘zgaruvchilarni o’rinlarini almashtirganda funksiyaning qiymati o‘zgarmadi.
Quyidagi n noma’lumli n ta ko‘phadlar simmetrik ko‘phadlar deyiladi:

$\sigma_1 = x_1 + x_2 + \ldots + x_n$

$\sigma_2 = x_1x_2 + x_1x_3 + \ldots + x_{n-1}x_n$

$\sigma_3 = x_1x_2x_3 + x_1x_2x_4 + \ldots + x_{n-2}x_{n-1}x_n$

...

$\sigma_{n-1} = x_1x_2\ldots x_{n-1} + x_1x_2\ldots x_{n-2}x_n + \ldots + x_2x_3\ldots x_n$

$\sigma_n = x_1x_2\ldots x_n$

Simmetrik ko‘phadlar mavzusiga oid bir nechta masalalarni yechamiz.
1. Ifodalarni asosiy (elementar) simmetrik ko‘phadlar orqali ifodalang:

$$\left(\frac{x_2}{x_1}+\frac{x_3}{x_2}+\frac{x_1}{x_3}\right)\left(\frac{x_1}{x_2}+\frac{x_2}{x_3}+\frac{x_3}{x_1}\right)$$

Yechish:
Umumiy maxrajga keltiramiz, qavslarni ochamiz va mos hadlarini ixchamlashtiramiz:

$$f(x_1,x_2,x_3) = 3x_1^2x_2^2x_3^2 + x_2^4x_1x_3 + x_2x_1^4x_3 + x_2x_1x_3^4 + x_2^3x_3^3 + x_1^3x_3^3 + x_1^3x_2^3$$

Bu yerda eng yuqori

$$\left(\frac{x_2}{x_1}+\frac{x_3}{x_2}+\frac{x_1}{x_3}\right)\left(\frac{x_1}{x_2}+\frac{x_2}{x_3}+\frac{x_3}{x_1}\right)=$$

$$=\frac{3x_1^2x_2^2x_3^2+x_2^4x_1x_3+x_2x_1^4x_3+x_2x_1x_3^4+x_2^3x_3^3+x_1^3x_3^3+x_1^3x_2^3}{(x_1x_2x_3)^2}=\frac{f(x_1,x_2,x_3)}{\sigma_3^2}$$

darajali yig‘indi quyidagi shaklda: $x_1^4x_2^1x_3^1$, eng yuqori daraja 4, darajalar yig'indisi 6 ga teng; yig‘indisi 6 ga teng, birinchisi 4 dan kichik, kamayish tartibida yozilgan uch darajali to'plamlarni ajratamiz:

4 1 1

3 3 0

3 2 1

2 2 2

4 2 0

U holda har bir $k_1k_2k_3$ ($k_1 \geq k_2 \geq k_3$) to‘plamga $\sigma_1^{k_1-k_2}\sigma_2^{k_2-k_3}\sigma_3^{k_3}$ ko‘phad mos qo‘yiladi. Quyidagilarni hosil qilamiz:

4 1 1: $\sigma_1^{4-1}\sigma_2^{1-1}\sigma_3^1=\sigma_1^3\sigma_3$

3 3 0: $\sigma_1^{3-3}\sigma_2^{3-0}\sigma_3^0=\sigma_2^3$

3 2 1: $\sigma_1^{3-2}\sigma_2^{2-1}\sigma_3^1=\sigma_1\sigma_2\sigma_3$

2 2 2: $\sigma_1^{2-2}\sigma_2^{2-2}\sigma_3^2=\sigma_3^2$

4 2 0: $\sigma_1^{4-2}\sigma_2^{2-0}\sigma_3^0=\sigma_1^2\sigma_2^2$

Tanlangan eng katta yig‘indi uchun koeffitsiyent 1 ga teng bo'lgani uchun funksiyani quyidagi ko'rinishda izlaymiz:

$$f(x_1,x_2,x_3)=1\sigma_1^3\sigma_3+B\sigma_2^3+C\sigma_1\sigma_2\sigma_3+D\sigma_3^2+E\sigma_1^2\sigma_2^2$$

B,C,D,E larni aniqmas koeffitsiyentlar usuli orqali topamiz:

1) $x_1=x_2=1$, $x_3=0$ bo‘lsin. U holda

$f(1,1,0) = 3\cdot1\cdot1\cdot0+1\cdot1\cdot0+1\cdot1\cdot0+1\cdot1\cdot0^4+1\cdot0+1\cdot0+1\cdot1=1$

$\sigma_1= x_1+x_2+ x_3=1+1+0=2$

$\sigma_2= x_1x_2+ x_1x_3+ x_2 x_3=1\cdot1+1\cdot0+1\cdot0=1$

$\sigma_3=x_1x_2x_3=1\cdot1\cdot0=0$

O‘rniga qo‘yamiz :

$$f(x_1,x_2,x_3)=\sigma_1^3\sigma_3+B\sigma_2^3+C\sigma_1\sigma_2\sigma_3+D\sigma_3^2+E\sigma_1^2\sigma_2^2$$

$$1=2^3\cdot0+B\cdot1+C\cdot2\cdot1\cdot0+D\cdot0+E\cdot2^2\cdot1^2$$

$$1=B+4E$$

2) $x_1= -1$, $x_2=1$, $x_3=0$ bo‘lsin.

U holda

$$f(-1,1,0)=-1$$

$$\sigma_1=0$$

$$\sigma_2=-1$$

$$\sigma_3=0$$

O‘rniga qo‘yamiz :

$$f(x_1,x_2,x_3)=\sigma_1^3\sigma_3+B\sigma_2^3+C\sigma_1\sigma_2\sigma_3+D\sigma_3^2+E\sigma_1^2\sigma_2^2$$

$$-1=-B$$

$$B=1$$

Quyidagi xulosani chiqarish mumkin:

$1 = B + 4E \Rightarrow 1 = 1 + 4E \Rightarrow E = 0$

$f(x_1, x_2, x_3) = \sigma_1^3\sigma_3 + \sigma_2^3 + C\sigma_1\sigma_2\sigma_3 + D\sigma_3^2$

3) $x_1 = 1$, $x_2 = 1$, $x_3 = 1$ bo‘lsin.

U holda

$f(1,1,1) = 9$

$\sigma_1 = 3,\ \sigma_2 = 3,\ \sigma_3 = 1$

O‘rniga qo‘yamiz :

$f(x_1, x_2, x_3) = \sigma_1^3\sigma_3 + \sigma_2^3 + C\sigma_1\sigma_2\sigma_3 + D\sigma_3^2$

$9 = 27 + 27 + 9C + D$

$9C + D = -45$

4) $x_1 = -1$, $x_2 = 1$, $x_3 = 1$ bo‘lsin.

U holda

$f(-1,1,1) = 1$

$\sigma_1 = 1,\ \sigma_2 = -1,\ \sigma_3 = -1$

O‘rniga qo‘yamiz :

$f(x_1, x_2, x_3) = \sigma_1^3\sigma_3 + \sigma_2^3 + C\sigma_1\sigma_2\sigma_3 + D\sigma_3^2$

$1 = -1 - 1 + C + D$

$C + D = 3$

C va D larni aniqlaymiz:

$$\begin{cases} 9C + D = -45 \\ C + D = 3 \end{cases} \Rightarrow \begin{cases} C = -6 \\ D = 9 \end{cases}$$

$f(x_1, x_2, x_3) = \sigma_1^3\sigma_3 + \sigma_2^3 - 6\sigma_1\sigma_2\sigma_3 + 9\sigma_3^2$ ni hosil qildik.

Javob: $\left(\frac{x_2}{x_1} + \frac{x_3}{x_2} + \frac{x_1}{x_3}\right)\left(\frac{x_1}{x_2} + \frac{x_2}{x_3} + \frac{x_3}{x_1}\right) = \frac{\sigma_1^3\sigma_3 + \sigma_2^3 - 6\sigma_1\sigma_2\sigma_3 + 9\sigma_3^2}{\sigma_3^2}$

Misol 2. $S(x_1^3x_2)$ monogen ko‘phadni elementar simmetrik ko‘phadlar orqali ifodalang.

Eslatma: bu turdagi monogen ko‘phad asosiy simmetrik ko‘phadlarni belgilash uchun analogi hisoblanadi, ya'ni bu holatda:

$S(x_1^3x_2) = x_1^3x_2 + x_2^3x_1 + x_3^3x_1 + x_1^3x_3 + x_2^3x_3 + \ldots + x_n^3x_{n-1}$

Algoritm avvalgi masalada berilganiga o'xshash, lekin o'zgaruvchilar sonini cheklamaydi. Ko'phadda qo‘shiluvchilar quyidagi ko'rinishga ega: $x_1^3x_2$, shuning uchun eng katta daraja 3 ga, darajalar yig'indisi esa 4 ga teng:

3 1 0 0 0: $\sigma_1^{3-1}\sigma_2^1 = \sigma_1^2\sigma_2$

2 1 1 0 0: $\sigma_1^{2-1}\sigma_2^{1-1}\sigma_3^1 = \sigma_1^1\sigma_3$

2 2 0 0 0: $\sigma_1^{2-2}\sigma_2^2 = \sigma_2^2$

1 1 1 1 0: $\sigma_1^{1-1}\sigma_2^{1-1}\sigma_3^{1-1}\sigma_4^1 = \sigma_4$

U holda bizning monogen ko‘phadimiz noaniq koeffitsiyentli ko'rinishda yozilishi mumkin, bunda eng katta qo‘shiluvchi 1 ga teng koeffitsiyentga ega:

$S(x_1^3x_2) = f(x_1, x_2, x_3, x_4, \ldots) = \sigma_1^2\sigma_2 + A\sigma_1\sigma_3 + B\sigma_2^2 + C\sigma_4$

Koeffitsiyentlarni topaylik:

1) $x_1 = x_2 = 1,\ x_3 = x_4 = ... = 0$ bo‘lsin.

$$f(1,1,0,0,...,0) = x_1^3x_2 + x_2^3x_1 + x_3^3x_1 + x_1^3x_3 + x_2^3x_3 + ... + x_n^3x_{n-1} =$$
$$= 1+1+0+0+0+.....0 = 2$$
$$\sigma_1 = x_1 + x_2 + ... + x_n = 1+1+0...+0 = 2$$
$$\sigma_2 = x_1x_2 + x_1x_3 + ... + x_{n-1}x_n = 1+0+...+0 = 1$$
$$\sigma_3 = x_1x_2x_3 + x_1x_2x_4 + ... + x_{n-2}x_{n-1}x_n = 0+0+...+0 = 0$$
$$\sigma_4 = x_1x_2x_3x_4 + x_1x_2x_4x_5 + ... + x_{n-3}x_{n-2}x_{n-1}x_n = 0+0+...+0 = 0$$

O‘rniga $S(x_1^3x_2) = \sigma_1^2\sigma_2 + A\sigma_1\sigma_3 - 2\sigma_2^2 + C\sigma_4$ qo‘yamiz:

$$2 = 2^2 \cdot 1 + A \cdot 2 \cdot 0 + B \cdot 1^2 + C \cdot 0$$
$$2 = 4 + B$$
$$B = -2$$

2) Va hosil qilamiz: $x_1 = x_2 = x_3 = 1,\ x_4 = x_5 = ... = 0$

$$f(1,1,1,0,...,0) = x_1^3x_2 + x_2^3x_1 + x_3^3x_1 + x_1^3x_3 + x_2^3x_3 + ... + x_n^3x_{n-1} =$$
$$= 1+1+1+1+1+1+0.....0 = 6$$
$$\sigma_1 = x_1 + x_2 + ... + x_n = 1+1+1+0...+0 = 3$$
$$\sigma_2 = x_1x_2 + x_1x_3 + ... + x_{n-1}x_n = 1+1+1+0+...+0 = 3$$
$$\sigma_3 = x_1x_2x_3 + x_1x_2x_4 + ... + x_{n-2}x_{n-1}x_n = 1+0+...+0 = 1$$
$$\sigma_4 = x_1x_2x_3x_4 + x_1x_2x_4x_5 + ... + x_{n-3}x_{n-2}x_{n-1}x_n = 0+0+...+0 = 0$$

O‘rniga qo‘yamiz:

$$6 = 3^2 \cdot 3 + A \cdot 3 \cdot 1 - 2 \cdot 9^2 + C \cdot 0$$
$$6 = 27 + 3A - 18$$
$$A = -1$$

Va hosil qilamiz: $S(x_1^3x_2) = \sigma_1^2\sigma_2 - \sigma_1\sigma_3 - 2\sigma_2^2 + C\sigma_4$

3) $x_1 = x_2 = x_3 = x_4 = 1,\ x_5 = ... = 0$

$$f(1,1,1,1,0...,0) = x_1^3x_2 + x_2^3x_1 + x_3^3x_1 + x_1^3x_3 + x_2^3x_3 + ... + x_n^3x_{n-1} =$$
$$= 12$$
$$\sigma_1 = x_1 + x_2 + ... + x_n = 1+1+1+1+0...+0 = 4$$
$$\sigma_2 = x_1x_2 + x_1x_3 + ... + x_{n-1}x_n = 6$$
$$\sigma_3 = x_1x_2x_3 + x_1x_2x_4 + ... + x_{n-2}x_{n-1}x_n = 4$$
$$\sigma_4 = x_1x_2x_3x_4 + x_1x_2x_4x_5 + ... + x_{n-3}x_{n-2}x_{n-1}x_n = 1$$

O‘rniga qo‘yamiz:

$$12 = 4^2 \cdot 6 - 1 \cdot 4 \cdot 4 - 2 \cdot 6^2 + C$$
$$C = 4$$

Va javobni hosil qilamzім: $S(x_1^3x_2) = \sigma_1^2\sigma_2 - \sigma_1\sigma_3 - 2\sigma_2^2 + 4\sigma_4$

Algebraik ifodalar. Algebraik ifoda deb algebraik amallar (qo‘shish, ayirish, ko‘paytirish, bo‘lish, darajaga ko‘tarish, ildiz chiqarish) bilan bog‘langan harflar va sonlardan tashkil topgan ifodaga aytiladi.

$x, y, \cdots, z$ kattaliklarni o'z ichiga olgan algebraik ifodani $A(x, y, \cdots, z)$ kabi yozamiz. Berilgan algebraik ifoda ko'rib chiqiladigan to'plam oldindan ko'rsatilishi kerak, ya'ni $x,\ y, \cdots,\ z$ kattaliklar qabul qilishi mumkin bo'lgan qiymatlar to'plami - butun, haqiqiy yoki kompleks va h.o. Agar maxsus aytilmagan bo'lsa, keyinchalik algebraik ifodalarni haqiqiy sonlar to'plamida ko'rib chiqamiz.

$A(x, y, \cdots, z)$ algebraik ifodada barcha algebraik amallarni bajarish mumkin bo'lgan kattaliklarning qiymatlariga - qabul qilish mumkin bo‘lgan qiymatlar deb ataymiz. Ular qabul qilish mumkin bo‘lgan qiymatlar sohasini (QQMBQS) tashkil qiladi.

Masalan, $\frac{1}{xy}$ algebraik ifodada qabul qilish mumkin bo‘lgan qiymatlar sohasini shunday $x, y \quad R$ juftliklar tashkil etadiki, $x \quad 0,\ y \quad 0$.

Avvalroq (§3 da) ayniyatga ta’rif berilgan edi. Ta’kidlash kerakki, ayniyatni $\equiv$ belgisi bilan belgilaymiz.

Masalan, $(x+y)^2 \quad x^2 + 2xy + y^2$ yoki $(x+y)(x-y) \quad x^2 - y^2$.

$A(x, y, \cdots, z)$ algebraik ifodadan $B(x, y, \cdots, z)$ ayniy algebraik ifodaga o'tish ayniy almashtirish deyiladi.

Algebraik ifodalar ratsional va irratsional algebraik ifodalarga bo'linadi.

Ta'rif 2. Agar algebraik ifodada ushbu ifodadagi biror kattalikka nisbatan faqat qo'shish, ayirish, ko'paytirish, bo'lish va butun darajaga ko'tarish amallari bajarilsa, unga ratsional ifoda deyiladi.

Masalan, $a + x - x^2$, $\frac{x^3 - xy}{x^2 - y^2 + 1}$ - ratsional ifodalar.

Ta'rif 3. Agar algebraik ifoda ushbu ifodadagi biror kattalikka nisbatan irratsional bo’lsa irratsional ifoda deyiladi.

Bu ta'rif shuni anglatadiki, biror kattalikka nisbatan irratsional bo'lgan ifodada bu kattalik ushbu ifodani mumkin bo'lgan soddalashtirilgandan keyin ham ildiz belgisi ostida qoladi.

Masalan, $\sqrt[3]{x^3}$ - x ga nisbatan ratsional ifoda, chunki $\sqrt[3]{x^3} = x$.

$\sqrt{x+1}$, $y\sqrt{x} + y^2 \quad \sqrt[3]{x^2}$ lar esa x ga nisbatan irratsional ifodalar, ikkinchisi y ga nisbatan ratsional.

Ratsional ifodalar butun va kasr ifodalarga bo‘linadi.

4-bobda qaralgan ko‘phadlar butun ratsional ifodalar hisoblanadi.

Kasr ratsional ifodalar yoki ratsional kasrlar deb ikki ko‘phadning nisbatiga aytiladi:

$$\frac{P(x, y, \cdots, z)}{Q(x, y, \cdots, z)} \quad (2)$$

<u>Kasr ratsional ifodalar va ular ustida amallar.</u> (2) ko‘rinishdagi kasrlarga ratsional kasrlar deyiladi. Quyidagilar ratsional kasrlarga misollar bo‘la oladi:

$\frac{x+y}{x^2 - xy + y^2}$; $\frac{3}{m^2 - n^2}$ va h.o.

Kasrning maxraji nolga teng bo‘lmagan qiymatlari – ratsional kasrning qabul qilishi mumkin bo‘lgan qiymatlari hisoblanadi.

Ratsional kasrlar uchun oddiy kasrlar uchun mavjud bo'lgan barcha xossalar qanoatlantiriladi. Oddiy kasrlar ustidagi amallar ratsional kasrlarda ham bajariladi, ya'ni ularni qo'shish va ayirish, ko'paytirish va bo'lish mumkin. Bir nechta misollarni ko'rib chiqamiz.

Misol 1. $\frac{x}{4a^3b}+\frac{5}{6ab^4}$ yig'indini toping.

Yechish. $12a^3b^4$ birhad - kasrlarning umumiy maxraji hisoblanadi. $3b^3$ va $2a^2$ lar - kasrlarning suratlari uchun qo'shimcha ko'paytuvchilar hisoblanadi. Hisoblaymiz:

$$^{3b^3/}\frac{x}{4a^3b}+{}^{2a^2/}\frac{5}{6ab^4}=\frac{3b^3x+10a^2}{12a^3b^4}.$$

Misol 2. $\frac{a+3}{a^2+ab}-\frac{b-3}{ab+b^2}$ ayirmani toping.

Yechish. $(a+b)ab$ ifoda - kasrlarning umumiy maxraji hisoblanadi. b va a lar - kasrlarning suratlari uchun qo'shimcha ko'paytuvchilar hisoblanadi. Hisoblaymiz:

$$\frac{a+3}{a^2+ab}-\frac{b-3}{ab+b^2}={}^{b/}\frac{a+3}{a(a+b)}-{}^{a/}\frac{b-3}{b(a+b)}=\frac{ab+3b-ab+3a}{ab(a+b)}=\frac{3(a+b)}{ab(a+b)}=\frac{3}{ab}$$

Bir nechta ratsional kasrlarni qo'shganda umumiy maxraj quyidagicha topiladi: barcha maxrajlarni ko'paytuvchilarga ajratamiz (agar iloji bo'lsa), istalgan maxrajni tanlab, uni qolgan maxrajlardan yetishmayotgan ko'paytuvchilarga ko'paytiramiz.

Misol 3. $\frac{1}{2b-2a}+\frac{1}{2b+2a}+\frac{a^2}{a^2b-b^3}$ yig'indini toping.

Yechish. Maxrajlarni $2(b-a)$, $2(b+a)$, $b(a-b)(a+b)$ ko'paytuvchilarga ajratamiz. Masalan, $2(b-a)$ birinchi maxrajni olamiz. Uni ikkinchi maxrajdagi $(b+a)$ ga hamda uchinchi maxrajdagi $-b$ ga ko'paytiramiz. Natijada, $2(b-a)(b+a)(-b)$ ko'rinishdagi umumiy maxrajni hosil qilamiz. Shuning uchun:

$$\frac{1}{2b-2a}+\frac{1}{2b+2a}+\frac{a^2}{a^2b-b^3}={}^{(a+b)(-b)/}\frac{1}{2(b-a)}+{}^{(b-a)(-b)/}\frac{1}{2(b+a)}+{}^{2/}\frac{a^2}{(b-a)(a+b)(-b)}=$$

$$=\frac{-ab-b^2+ab-b^2+2a^2}{2(b-a)(b+a)(-b)}=\frac{-2b^2+2a^2}{-2(b-a)(b+a)b}=\frac{-2(b^2-a^2)}{-2b(b^2-a^2)}=\frac{1}{b}.$$

Kasrlarni ko'paytirish, bo'lish va darajaga ko'tarish oddiy kasrlar bilan bir xil qoidalarga muvofiq amalga oshiriladi:

a) $\frac{A}{B}\ \frac{C}{D}=\frac{AC}{BD}$, $B\quad 0$; $D\quad 0$; b) $\frac{A}{B}:\frac{C}{D}=\frac{A}{B}\ \frac{D}{C}$ $(B, C, D\quad 0)$;

d) $\frac{A}{B}^{m}=\frac{A^m}{B^m}$, $B\quad 0$; e) $(A^m)^n=A^{mn}$.

Xulosa:

Elementar simmetrik ko'phadlar, ratsional ifodalar va ularni soddalashtirish. Elementar simmetrik ko'phadlar: Bu ko'phadlar bir yoki bir nechta o'zgaruvchilar bo'yicha simmetrik xususiyatlarga ega bo'lib, ularning koeffitsiyentlari simmetrik funktsiyalarni ifodalaydi. Masalan, uchta o'zgaruvchi uchun simmetrik ko'phad $a_1x_1+a_2x_2+a_3x_3$ shaklida bo'lishi mumkin.

1. Ratsional ifodalar: Ratsional ifodalar, ikki polinomning nisbatidan iborat bo'lib, R(x)=$\frac{Px}{Qx}$ ko'rinishida ifodalanadi. Bu ifodalar algebraik manipulyatsiyalarni osonlashtiradi va masalalarni yechishda keng qo'llaniladi..

2. Soddalashtirish: Ratsional ifodalarni soddalashtirish, odatda, umumiy omillarni ajratish yoki polinomlarni omillarga ajratish orqali amalga oshiriladi. Bu jarayon ifodani osonroq tushunishga va hisoblashlarga yordam beradi.
3. Amaliy qo‘llanish: Elementar simmetrik ko‘phadlar va ratsional ifodalar algebraik tenglamalarni yechishda, fizik muammolarni modellashtirishda va matematik tahlilning boshqa sohalarida qo‘llaniladi.

Natijada, elementar simmetrik ko‘phadlar va ratsional ifodalar algebraik nazariyada muhim o‘rin tutadi, ular orqali ko‘plab matematik masalalarni oson va samarali hal etish mumkin.

Adabiyotlar ro’yxati:

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